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High-Reynolds-number viscous flow in collapsible tubes

By O. R. TUTTY
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW

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This study is concerned with steady laminar high-Reynolds-number flow in collapsible tubes, where the position of the tube wall is a function (the tube law) only of the pressure exerted by the fluid on the wall. The system is controlled by two main parameters: the Reynolds number of the incoming flow, and the 'compliance', which characterizes the response of the wall to a change in fluid pressure. Restrictions are placed on these parameters so that the streamwise lengthscale is large, and, to the order worked, the pressure is uniform across the tube. Attention is restricted to (axi)symmetric systems.

Channels are considered in most detail, the results for axisymmetric pipes being largely similar.

For a model tube law the flow in the converging section of a channel is investigated in detail. Solutions are presented for certain of the parameter values. For some channels a singularity is found in the solution such that the channel width tends to zero at a finite distance downstream. No way was found to integrate past this singularity.

For particular channels and pipes, solutions are found only for flows in which the mean fluid velocity is less than the propagation speed of frictionless waves. This is consistent with experimental results.

1. Introduction

It is thought that fluid-dynamical properties, particularly the local shear stress, can have important physiological implications for the flow in the blood vessels of animals (see e.g. Caro, Fitz-Gerald & Schroter 1969, 1971; Lighthill 1972), and recently attention has been given to flow in non-uniform tubes under a variety of conditions (e.g. Jaffrin & Shapiro 1971; Wild, Pedley & Riley 1977; Shapiro 1977*a*; Secomb 1978; Pedley 1980). We are interested in the behaviour of collapsible fluid–tube systems, which could be regarded as idealized models of physiological systems, such as the flow in veins, the urethra, and the pulmonary airways (for a more detailed list see Shapiro 1977*b*).

We consider steady high-Reynolds-number flow in an infinitely long collapsible tube where the position of the wall is known as a function of the pressure exerted by the fluid on the wall alone. We assume that far upstream the tube has a constant cross-sectional area and shape, and that there is a known reference pressure above which the tube takes this rigid shape and below which the position of the wall can vary. Also, we assume that the 'tube law', the relationship between the position of the wall or the cross-sectional area and the internal pressure, is a continuous function. Hence the position of the wall will be continuous everywhere the fluid pressure at

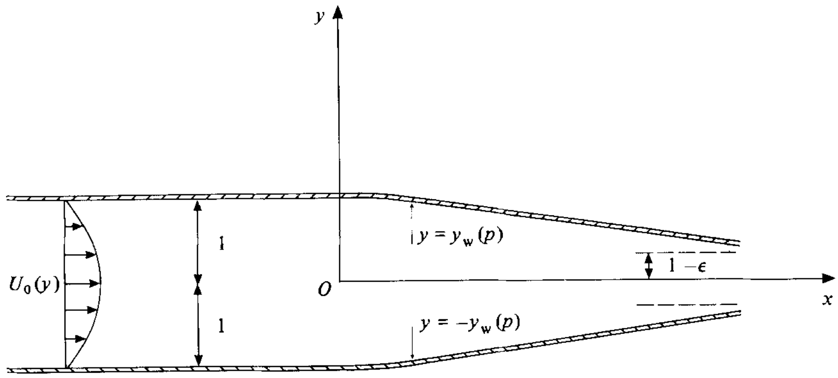

FIGURE 1. The geometry and coordinates for a collapsible channel.

the wall is continuous. The reasons for and validity of using a tube law are discussed briefly below.

We are concerned only with (axi)symmetric-tube fluid systems in this study (some aspects of non-symmetric systems will be considered in a later study), and consider only converging tubes. In general the results for analogous symmetric channels and axisymmetric pipes are similar, and usually we consider the channel problem. The significant differences between pipe and channel flows are discussed in §6 below.

Equations for a collapsible channel

The fluid is incompressible with steady laminar motion. Let ρ be the fluid density, a be the half-width of a channel when in its rigid upstream form, and U^* be a characteristic velocity far upstream. The channel problem is two-dimensional, and we write the pressure as $\rho U^{*2}p$ and the velocity as $U^*(u,v)$ in Cartesian coordinates (x,y), where ax and ay are the distances downstream and across the channel respectively The origin is taken as the centre of the channel at the point where the walls first deviate from the rigid upstream form (see figure 1). The reference pressure at which the collapse starts is taken to be zero.

The properties of the two channel walls are taken to be identical. Hence the flow is in general expected to be symmetric. In the problems studied below, the pressure is independent of y. Thus the position of the walls is given by $y = \pm y_w$, where $y_w = y_w(p)$ is the given tube law. The boundary conditions are those of no slip on the walls and matching to fully developed Poiseuille flow upstream:

$$\left.\begin{aligned} u = v = 0 \quad &\text{at} \quad y_w = \pm y_w(p), \\ u \to U_0(y) \quad \text{and} \quad v \to 0 \quad &\text{as} \quad x \to \infty, \end{aligned}\right\} \tag{1.1}$$

where $U_0(y) = \frac{1}{2}(1-y^2)$. If the flow is symmetric then no slip at one of the walls can be replaced by a symmetry condition at $y = 0$. In practice, since no upstream influence is found, the upstream condition can be applied at $x = 0$.

We use a tube law of the form

$$y_w(p) = 1 - \epsilon S(\mu p), \tag{1.2}$$

where $\mu > 0$ and $0 < \epsilon \leqslant 1$ are non-dimensional constants, and $S(t)$ is a continuous monotonic function such that $S(t) = 0$ when $t \geqslant 0$, $S(t) > 0$ when $t < 0$, $S(t)$ is $O(1)$ when $-t$ is $O(1)$, and $S(t) \to 1$ as $t \to -\infty$. Thus the rigidity of the wall whenever the pressure is positive is assured. The form of the tube law is discussed below. If $\epsilon = 1$ we say that the tube undergoes a complete collapse, and if $0 < 1-\epsilon \ll 1$ a severe collapse.

Our interest is in high-Reynolds-number flow, and we assume

$$Re = U^* a/\nu \gg 1 \tag{1.3}$$

throughout, where ν is the kinematic viscosity of the fluid. In addition to (1.3), restrictions will be placed on ϵ and μ so that, to the order considered, the transverse momentum equation implies that the pressure is independent of y.

So far we have not considered the lengthscales to be used. For flow in a tube with a fixed constriction, the lengthscales are obtained from the known dimensions of the constriction (see e.g. Smith 1976*a*,*b*, 1978, 1979). We have, in a sense, the opposite situation; that is, it is the pressure distribution that determines the position of the walls, and it could be said that our major task is to follow the development of the pressure from its value in the incoming Poiseuille flow. Accordingly, we assume that sufficiently far upstream the pressure gradient is $O(Re^{-1})$ and use this to find initial lengthscales for any given situation. This gives large $(O(Re/\mu))$ axial lengthscales for all the problems considered here. In particular, if $\mu = O(1)$, the governing equations are

$$\left.\begin{aligned} \frac{\partial u}{\partial X}+\frac{\partial V}{\partial y} &= 0, \\ u\frac{\partial u}{\partial X}+V\frac{\partial u}{\partial y} &= -\frac{\mathrm{d}p(X)}{\mathrm{d}X}+\frac{\partial^2 u}{\partial y^2}, \end{aligned}\right\} \tag{1.4}$$

the nonlinear boundary-layer equations, where $X = Re^{-1}x$ and $V = Re\,v$.

The tube law

In common with a number of other studies of flow in collapsible tubes (e.g. Wild *et al.* 1977; Shapiro 1977*a*; Kamm & Shapiro 1979), we use an algebraic form of the tube law when it is explicitly required. Various theoretical laws of this form have been proposed, both for distended elastic tubes with a circular cross-section (e.g. Taylor & Gerrard 1977) and for buckled tubes with negative transmural pressure (e.g. Flaherty, Keller & Rubinow 1972). Strictly, these relationships apply to tubes filled with a stationary fluid that are distended by a uniform transmural pressure. If the fluid is viscous and in motion, the pressure at the wall is unlikely to be uniform, and there will be a shearing force on the inner wall. Further, in any rigorous analysis the effects of the bending stiffness of the wall and the conditions at the outer surface of the wall must be considered. Hence it may be necessary to place severe restrictions on the fluid–tube system to justify this assumption. In particular, the effect of the longitudinal tension in the tube wall must be negligible, and we must have a long axial lengthscale (see Tutty 1982; Cowley 1982). In all the cases considered here the latter of these conditions is satisfied. Note that, even if the use of a tube law is valid, the correct law may not be that obtained from a uniformly distended elastic tube (Cowley 1982).

Where the tube law is required explicitly, we use

$$S(\mu p) = \begin{cases} 0 & (p \geqslant 0), \\ 1-[1+(\mu p)^2]^{-\frac{1}{2}q} & (p < 0), \end{cases} \tag{1.5}$$

where μ and q are positive non-dimensional constants. This relationship has a similar general form to those derived for certain elastic tubes (Taylor & Gerrard 1977;

Flaherty *et al.* 1972), and those used in other studies of flow in collapsible tubes (e.g. Shapiro 1977*a*). Writing $\mu = \bar{\mu}\rho U^{*2}$, we see that there are three parameters, $\bar{\mu}$, q and ϵ, that characterize the elastic response of the tube to changes in the pressure $\rho U^{*2}p$, and that, unlike q and ϵ, μ is a composite parameter with its value depending on both the incoming flow and the elastic properties of the tube. An important feature of (1.5) is that $\mathrm{d}S/\mathrm{d}p = 0$ at $p = 0$. This ensures a smooth transition between the 'rigid' and 'collapsible' sections of the tube ($p > 0$ and $p < 0$ respectively), which eliminates any 'entrance effects' which might arise from a discontinuous change in the slope of the walls where $p = 0$. Also, such a discontinuous change could invalidate the use of a tube law. Since for a given tube (values of q and ϵ) the rate of change of cross-sectional area with respect to the pressure is determined by μ, we call a tube–fluid system with $\mu \gg 1$, $\mu = O(1)$ or $\mu \ll 1$ a system with a high, moderate or low compliance respectively. Note that if $(\mu p)^2$ in (1.5) is replaced by $(\mu p)^m$, where $m > 1$, then an analysis similar to that below could be carried out, with essentially the same results.

Contents

This paper is concerned with symmetric channels and axisymmetric pipes with a low or moderate compliance (complementary papers dealing with channels with a high compliance and non-symmetric pipes are in preparation). We consider 'substantial' collapses only (i.e. those with $\epsilon = O(1)$). Tutty (1982) gives solutions for 'fine' collapses ($\epsilon \ll 1$). Section 2 describes the general structure of the flow, with channels obeying (1.5) with $q \leqslant \frac{1}{3}$, $\frac{1}{3} < q < \frac{1}{2}$, $q = \frac{1}{2}$ and $q > \frac{1}{2}$ studied in detail in §§3–6 respectively. The flow is Poiseuille-like if $q < \frac{1}{2}$. A singularity is found such that the channel collapses completely at a finite distance downstream if $\frac{1}{3} < q \leqslant \frac{1}{2}$. The flow structure derived does not appear to break down as the singularity is reached (§5). We found no solution for $q > \frac{1}{2}$ (§6) or for $\mu \geqslant 18$ with $q = \frac{1}{2}$ (§5) when the collapse was complete. Channels with low compliance are considered in §7. Section 8 deals with axisymmetric pipes. Again, solutions are found only for tube laws where the cross-sectional area A varies as $(-p)^{-m}$ with $m \leqslant \frac{1}{2}$. Also a limit of $\mu \leqslant 128$ applies to the solution found for $m = \frac{1}{2}$. Our results are compared with experimental results in §9, where the limits on our solutions are found to be consistent with experimentally observed phenomena. Some remarks concerning the results are made in §10.

The nonlinear problem defined by (1.1)–(1.5) must be solved numerically. We used the 'Keller-box' method, as detailed by Smith (1974) for the interactive boundary-layer equations. In particular, the (non-asymptotic) solutions presented in figures 3, 4 and 8 were obtained in this way. Some of our solutions were checked using a 'stream-function–vorticity' formulation centred on the half-step in x with a Gauss–Seidel iteration. Good agreement was found between the two methods.

Throughout this paper the stream function is defined in the standard way.

2. General structure of the flow

Consider a complete collapse ($\epsilon = 1$) with a moderate compliance ($\mu = O(1)$). Close to the origin ($0 < X \ll 1$; $-\mu p \ll 1$), the flow is given to leading order by Poiseuille flow with a perturbation to the stream function $O(X^{\frac{7}{3}})$. This perturbation has an inviscid rotational form in the core (region II, figure 2), with viscous layers of width $O(X^{\frac{1}{3}})$ at the walls (region I, figure 2). The powers of X are forced by the form of the tube law, the no-slip condition and the (necessary) balance of viscous and inertial forces in the wall layers. We omit the details (which can be found in Tutty 1982). Clearly there is a fairly smooth change in the flow and the wall position near the origin.

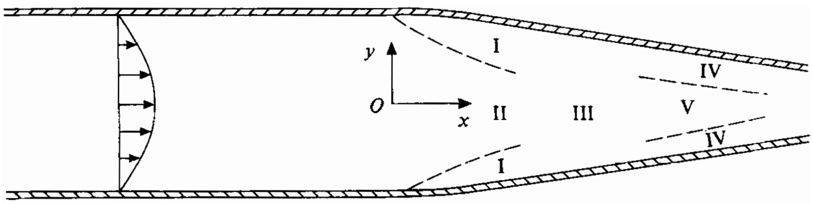

FIGURE 2. Flow regions.

The flow structure outlined above is valid only for $1-y_w \ll 1$, and not when $X = O(1)$, or more precisely when $-\mu p = O(1)$, and the walls have moved a finite distance into the channel (region III, figure 2). When this occurs, $u - U_0(y)$ is also $O(1)$, and the flow is governed by the nonlinear boundary-layer equations (1.4). Numerical methods must be used to find the flow in this region.

Further downstream, where $-\mu p \gg 1$ (regions IV and V, figure 2), the tube law (1.5) implies

$$y_w = k(-p)^{-q} \tag{2.1}$$

to leading order, where $k = \mu^{-q}$ is $O(1)$. As mass conservation requires that the average streamwise velocity is $O(y_w^{-1})$, the ratio of pressure gradient to inertial forces is $O(y_w^{2-1/q})$. Hence as $p \to -\infty$ and $y_w \to 0$ the pressure gradient will dominate the inertia if $q < \frac{1}{2}$ and the inertia the pressure force if $q > \frac{1}{2}$. If $q = \frac{1}{2}$ there will be a balance between these forces as the channel collapses. For $q < \frac{1}{2}$ there must eventually be a balance between the pressure gradient and the viscous forces (which are $O(y_w^{-3})$) in the fluid. This implies a lubrication solution, and, with an appropriate change of origin, that $X = O(y_w^{3-1/q})$ as $y_w \to 0$. It follows that for $q < \frac{1}{3}$ the collapse will extend indefinitely far downstream, while, for $\frac{1}{3} < q < \frac{1}{2}$, y_w will tend to zero at a finite value (X_0) of X. For $q = \frac{1}{3}$ it is found that the channel collapses exponentially, i.e. at the fastest rate consistent with a collapse extending to infinity.

3. Channels with $q \leqslant \frac{1}{3}$

First suppose that $q < \frac{1}{3}$, $\epsilon = 1$ and $\mu = O(1)$. As $p \to -\infty$ viscous diffusion dominates the inertia, and the flow is given to leading order by the self-similar structure

$$u = G(x) f(\zeta), \tag{3.1a}$$

$$p = -p_0 G^{1/q}, \tag{3.1b}$$

where $G(X) = X^{q/m}$, $\zeta = yG$, $m = 1-3q > 0$, p_0 is a constant, and an implicit origin shift (which must be determined numerically) has been included. The solution is easily found to be Poiseuille flow:

$$f(\zeta) = \frac{p_0}{2m}(\zeta_w^2 - \zeta^2), \tag{3.2}$$

where $p_0 = (m/k^3)^{1/m}$ and $\zeta_w = kp_0^{-q}$ gives the position of the upper wall. The skin friction is

$$\tau = -\frac{p_0^{1-q}}{m} X^{2q/m}.$$

As is usual with self-similar solutions, we have not used any initial conditions. Note that, although (3.2) is consistent with the incoming Poiseuille flow, lubrication theory does not apply everywhere, and there is a region (III, figure 2) where inertia is important at leading order.

The powers of X in (3.1) depend only on the value of q, and as might be expected an increase in q brings (with respect to X) an increase in the order of the pressure, velocity and skin friction, and a decrease in the negative order of the channel width. As $q \to \frac{1}{3}$, $q/m \to \infty$, which suggests formally an exponential structure when $q = \frac{1}{3}$. Examination of the numerical solution of (1.4) and (1.1) showed that the effect of increasing q and μ was to steepen the main part of the collapse and shift it closer to the origin, with the opposite effect from decreasing q and μ. This is consistent with the predicted faster response of the system with larger values of q and/or μ.

With $q < \frac{1}{3}$ the complete collapse admitted by $\epsilon = 1$ will be achieved only in the limit as $X \to \infty$, and the analysis above is valid far downstream.

Let us assume now that the collapse is severe but not complete, i.e. $0 < 1-\epsilon \ll 1$. Then (3.1) is valid until y_w is $O(1-\epsilon)$, when the problem must be rescaled. Let

$$u = (1-\epsilon)^{-1}\,\bar{u}(\bar{X}, \bar{Y}) \quad \text{and} \quad p = (1-\epsilon)^{-1/q}\,\bar{p}(\bar{X}),$$

where

$$y = (1-\epsilon)\,\bar{Y} \quad \text{and} \quad x = Re\,(1-\epsilon)^{-m/q}\,\bar{X}.$$

To leading order the upper wall is given by $\bar{Y}_w = 1+\epsilon k(-\bar{p})^{-q}$. The flow is still viscous here, the boundary conditions are no slip at the wall and a mass-conservation condition, and the solution for $\bar{u}$ has the lubrication form (3.1a) and (3.2) with $G(\bar{X}) = \bar{Y}_w^{-3}, p_0 = m = 1$ and $\zeta = \bar{Y}$. The pressure can be found from $\bar{p}' = -\bar{Y}_w^{-3}$, where in practice the starting point for the integration must be found numerically. We note that a similar pressure relationship holds upstream where (3.1) is valid. It is easily shown that this solution matches with (3.1) and (3.2) upstream, and that it takes the expected form far downstream, i.e. Poiseuille flow in the limit as $\bar{X} \to \infty$ and $\bar{Y}_2 \to 1$. For $-\bar{p} \gg 1$ it can be expanded in negative powers of $\bar{X} \gg 1$. We omit the details.

Suppose now that $q = \frac{1}{3}$ and $\epsilon = 1$. If $-\mu p \gg 1$ then to leading order the tube law is given by (3.1), and the necessary balance between the viscous force and the pressure gradient in the fluid implies that $X = O(\ln y_w)$ as $y_w \to 0$. Thus the collapse has an exponential form, as predicted. The flow, which is viscous in character, is given by (3.1) and (3.2) with $G(X) = e^{aX}$, $p_0 = 1$, $m = k^3$, $\zeta = y\,e^{aX}$ and $a = 1/(3k^3)$. The walls are given by $\zeta = \pm\zeta_w = \pm k$, and an implicit origin shift has been included. Again (cf. $q < \frac{1}{3}$) a larger μ results in a faster collapse and a smaller μ a slower collapse. For a severe collapse the analysis proceeds as for $q < \frac{1}{3}$ above.

Figure 3 shows results for various q with $\epsilon = \mu = 1$. The asymptotic solution given for $-\mu p \gg 1$ are not shown as they are graphically indistinguishable from the numerical solution in the region where they are valid. However, we note that for $q = \frac{1}{3}$ the pressure and skin-friction curves take the predicted form, i.e. straight lines on a logarithmic scale except near the origin.

4. Channels with $\frac{1}{3} < q < \frac{1}{2}$

Consider now a complete collapse ($\epsilon = 1$), a moderate compliance ($\mu = O(1)$) and $\frac{1}{3} < q < \frac{1}{2}$. We know from §2 that where the tube law takes the form (2.1) (regions IV and V figure 2), the pressure gradient will dominate the inertial forces in the fluid as $p \to -\infty$, and that the width of the channel tends to zero at a finite value X_0 of X. That is, as $y_w \to 0$ the flow is viscous-dominated and acts as if there is a sink at $X = X_0$. For $0 < X_0 - X \ll 1$ the flow has a self-similar structure, viz (3.1) and (3.2) with $G(X) = (X_0-X)^{-q/m}$, $\zeta = yG$, $m = 3q-1 > 0$, $p_0 = (m/k^3)^{-1/m}$, and the tube law

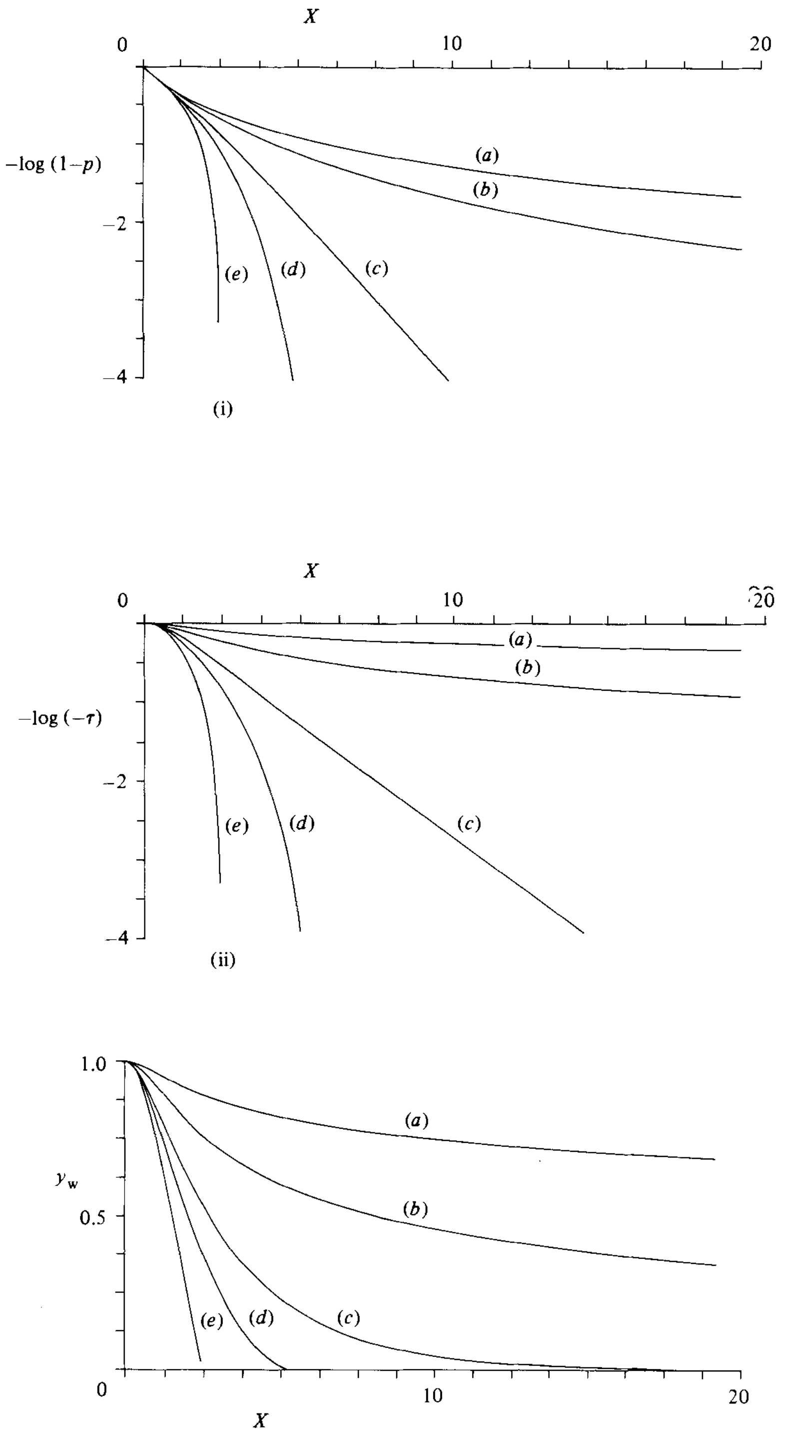

FIGURE 3. (i) Pressure p (on a logarithmic scale). (ii) Upper-wall skin friction τ (on a logarithmic scale). (iii) Channel half-width y_w. Results against distance for a channel with $\epsilon = \mu = 1$. The asymptotic solutions are graphically identical (where valid). (*a*) $q = 0.1$; (*b*) 0.2; (*c*) $\frac{1}{3}$; (*d*) 0.4, (*e*) $\frac{1}{2}$.

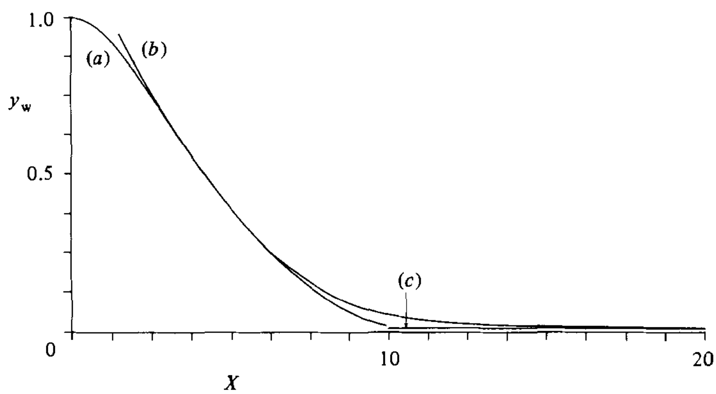

FIGURE 4. Half-width y_w against distance for a channel with $q = 0.4$, $\mu = 1$ and $\epsilon = 0.99$. From (a) numerical solution of the full problem (1.1), (1.2), (1.4) and (1.5); (b) self-similar solution valid when $y_w \sim k(-p)^{-0.4}$; (c) far-downstream analysis with $\bar{Y}_w \sim 1+\epsilon k\bar{X}^{-0.4}$.

has the local form $\zeta = \zeta_w = kp_0^{-q}$. To leading order, $y_w = \zeta_w(X_0-X)^{q/m}$, and since $q < \frac{1}{2}$ the slope of the walls tends to zero as $X \rightarrow X_0-$.

The pressure gradient is $p' = -y_w^{-3}$, with y_w from (2.1). Integrating

$$X-X_1 = \frac{1}{m\mu^{3/q}}[(-p_1)^{-m}-(-p)^{-m}],$$

where $X > X_1$ and $p_1 = p(X_1)$. It follows that

$$X_0 = X_1 + \frac{(-p_1)^{-m}}{m\mu^{3/q}}. \tag{4.1}$$

Rubinow & Keller (1972) assumed that Poiseuille flow is valid locally and produced a simple model of steady flow in a collapsible tube. Their model predicts that as the outlet transmural pressure tends to infinity the mass flow tends to an upper limit, a result which depends on the existence of a certain integral. For our problem this integral is

$$\int_{-\infty}^{0} y_w^{-3}(p)\,\mathrm{d}p = \int_{-\infty}^{p_1} y_w^{-3}(p)\,\mathrm{d}p + \int_{p_1}^{0} y_w^{-3}(p)\,\mathrm{d}p$$

which exists if $q > \frac{1}{3}$. In effect, Rubinow & Keller assume that a complete collapse to a given point (the outlet) is possible and calculate the mass flow consistent with this collapse. For our problem, this is equivalent to assuming that $q < \frac{1}{3}$ and that X_0, X_1 and p_1 are known, and hence calculating μ from (4.1). In contrast, we assume a particular mass flow in a given tube, i.e. values of μ and q, and calculate the point X_0 at which the channel collapses completely.

As before, an increase in q or μ steepens the main part of the collapse and moves it upstream.

The numerical solution of the problem with $\epsilon = 1$ clearly displays the predicted singularity in the solution, as can be seen in figure 3.

The boundary-layer approximation does not appear to break down for any $X_0 - X > 0$, and the flow structure given above remains valid until the limit is reached. Clearly, this is not physically reasonable (see §5 for discussion).

If the collapse is severe rather than complete then the analysis continues as for $q < \frac{1}{3}$ (§3). Figure 4 shows the channel half-width against X (obtained numerically

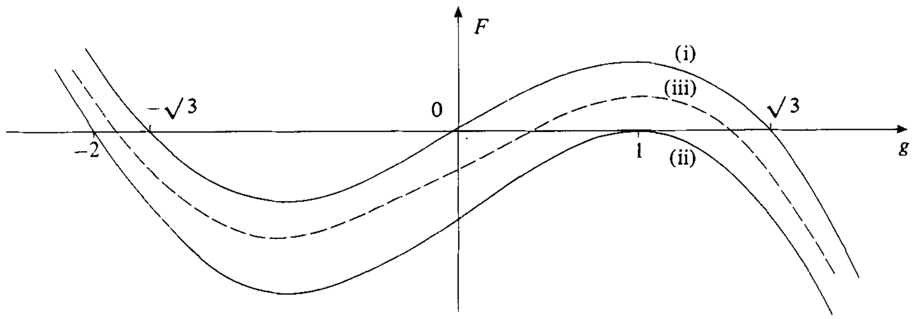

FIGURE 5. $F(g)$: (i) has $g_0 = 0$, $a = \sqrt{3}$ and $b = -\sqrt{3}$; (ii) has $g_0 = a = 1$ and $b = -2$; and (iii) has $g_0 = \frac{2}{3}$, $a = \frac{1}{5}(6\sqrt{2}-1)$ and $b = \frac{1}{5}(6\sqrt{2}+1)$. All curves with $0 < g_0 < 1$ lie between (i) and (ii), and unless $0 \leqslant g_0 \leqslant 1$ either $V(0) > 0$ or $V(g) < 0$ for all $g > 0$.

from (1.1), (1.2), (1.4) and (1.5)) with $\epsilon = 0.99$, $\mu = 1$ and $q = 0.4$. Also shown are the theoretical results for the main part of the collapse, and for far downstream where $\overline{Y}_w = 1 + \epsilon k \overline{X}^{-q} + O(\overline{X}^{-2q})$.

5. Channels with $q = \frac{1}{2}$

We now study the channel with $q = \frac{1}{2}$, $\mu = O(1)$ and $\epsilon = 1$. This is the largest value of q for which a complete steady solution has been found when $\mu = O(1)$ (see §6). Also, with $q = \frac{1}{2}$ there is an upper limit of $\mu < 18$ for which a steady solution of the form assumed exists (see below).

With $q = \frac{1}{2}$ the inertial force and pressure gradient in the fluid are of the same order of magnitude when the tube law is given by (2.1) (§2). Assuming a balance with the viscous force implies that y_w is $O(X_0 - X)$ for some finite X_0 as $X \to X_0-$. Transforming the stream function and pressure to

$$\left.\begin{aligned} \psi &= (2p_0)^{\frac{1}{4}} f(\zeta), \\ p &= -p_0(X_0 - X)^{-2}, \end{aligned}\right\} \tag{5.1}$$

where $\zeta = (2p_0)^{\frac{1}{4}}\, y(X_0 - X)^{-1}$, the streamwise momentum equation becomes

$$g'' + 1 - g^2 = 0, \tag{5.2}$$

where $g = f'$ gives the streamwise velocity component in the similarity solution. The tube law has the local form

$$\zeta_w = k(2/p_0)^{\frac{1}{4}}. \tag{5.3}$$

Conservation of mass and no slip at the walls require

$$\int_{-\zeta_w}^{\zeta_w} g(\zeta)\, \mathrm{d}\zeta = \tfrac{2}{3}(2p_0)^{-\frac{1}{4}}, \quad g = 0 \quad \text{at} \quad \zeta = \pm\zeta_w \tag{5.4}$$

in turn.

Equation (5.2) is similar to that for Jeffery–Hamel flow between non-parallel plane walls (Rosenhead 1940). Jeffery–Hamel flows are not in general unique. In fact, there is an infinite number of solutions, although the number that are valid can be restricted severely under certain conditions (Fraenkel 1962). There is also an infinite number of solutions to the present equation, as will now be shown.

Integrating (5.2) produces

$$\tfrac{1}{2}g'^2+F(g)=0, \tag{5.5a}$$

where

$$F(g)=-\tfrac{1}{3}(g_0-g)\,(a-g)\,(g-b), \tag{5.5b}$$

and g_0, $a=\frac{1}{2}[-g_0+(12-3g_0^2)^{\frac{1}{2}}]$ and $b=-\frac{1}{2}[g_0+(12-3g_0^2)^{\frac{1}{2}}]$ are the solutions of $g'=0$. If a, b and g_0 are all real let $b\leqslant g_0\leqslant a$, otherwise let b be the single real solution of $g'=0$.

We require $F(g)$ to be zero at $\zeta=\pm\zeta_{\mathrm{w}}$ and $F(g)<0$ in $-\zeta_{\mathrm{w}}<\zeta<\zeta_{\mathrm{w}}$. From inspection of the graph of $F(g)$ (figure 5), we see that the maximum possible value of g is unity, that $g\neq a$ unless $g_0=a=1$, and that there are two basic solutions, a completely forward flow with $0\leqslant g\leqslant g_0$, and completely reversed flow with $b\leqslant g\leqslant 0$, which can exist if $b\leqslant-\surd 3$. Clearly the completely reversed flow cannot satisfy mass conservation. However, if $0<g_0\leqslant 1$ there are composite flows with $g_{\max}=g_0$ and $g_{\min}=b$, which satisfy continuity and have regions of both forward and reversed flow, and our problem does not have a unique solution (see below).

First, we examine the completely forward flow in which the velocity is symmetric and is monotonic in the upper and lower halves of the channel. We replace no slip at the lower wall with $g'(0)=0$, which is automatically satisfied as $g(0)=g_0$ in (5.5). In the upper half of the channel g' is negative, and (5.5) and the no-slip condition imply that

$$\zeta_{\mathrm{w}}-\zeta=\sqrt{\frac{3}{2}}\int_0^g\frac{\mathrm{d}t}{[(g_0-t)\,(a-t)\,(t-b)]^{\frac{1}{2}}}. \tag{5.6}$$

At $\zeta=0$ this gives

$$k\left(\frac{2}{p_0}\right)^{\frac{1}{4}}=\sqrt{\frac{3}{2}}\int_0^{g_0}\frac{\mathrm{d}t}{[(g_0-t)\,(a-t)\,(t-b)]^{\frac{1}{2}}}, \tag{5.7}$$

which with the condition from conservation of mass, viz

$$\tfrac{1}{3}(2p_0)^{-\frac{1}{4}}=\sqrt{\frac{3}{2}}\int_0^{g_0}\frac{t\,\mathrm{d}t}{[(g_0-t)\,(a-t)\,(t-b)]^{\frac{1}{2}}}, \tag{5.8}$$

determines the values of g_0 and p_0 for a given value of the known parameter k $(=\mu^{-\frac{1}{2}})$. In practice it proves easier to take a particular value of g_0 and determine the values of k and p_0 consistent with this g_0. We note that (5.7) and (5.8) can be written in terms of elliptic integrals.

The solution for $1-g_0\ll 1$

Some important details of the local behaviour of the flow can be deduced from the solution when g_0 is close to unity, which we now examine in detail. It can be shown that $\zeta_{\mathrm{w}}\sim-(1/\surd 2)\ln(1-g_0)\to\infty$ and $k\to k_0$ as $g_0\to 1-$, where $k_0=1/(3\surd 2)$. The behaviour of g, and of the constants g_0, p_0 and ζ_{w}, as $k\to k_0$ can be deduced directly from the momentum equation and conservation of mass. Equation (5.2), the monotonicity of g in $0\leqslant\zeta\leqslant\zeta_{\mathrm{w}}$, and $\zeta_{\mathrm{w}}\gg 1$, imply that to leading order

$$g=1 \tag{5.9}$$

in the core where $\eta=\zeta_{\mathrm{w}}-\zeta\gg 1$ (V, figure 2). Further, the next term in the expansion for g in the core must be exponentially small in ζ_{w}, as must $1-g_0$. If for $\eta=O(1)$, i.e close to the wall (IV, figure 2), we write $g(\zeta)$ as $\hat{g}(\eta)$, then $\hat{g}(\eta)$ must satisfy (5.2), $\hat{g}(0)=0$ (no slip at the wall) and $\hat{g}\to 1$ as $\eta\to\infty$ (matching to the core flow). This Falkner–Skan problem is the same as that arising in the flow in a converging channel with intersecting plane walls. The solution is

$$\hat{g}(\eta)=3\tanh^2(\eta_0+\eta/\surd 2)-2, \tag{5.10}$$

where $\eta_0=\tanh^{-1}\surd\frac{2}{3}=1.146216$.

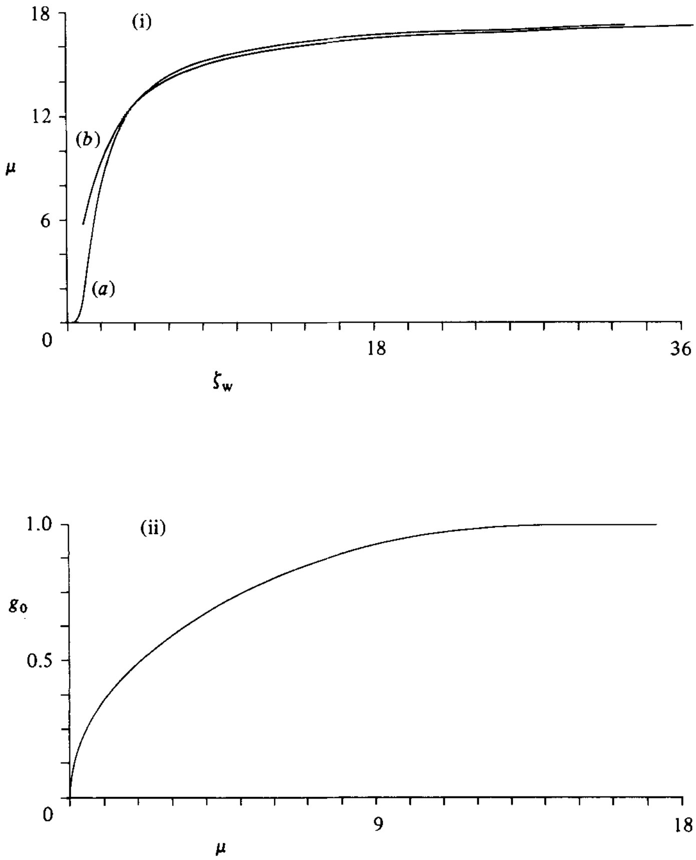

FIGURE 6. Coefficients of the asymptotic solution for $-\mu p \gg 1$ with $\epsilon = 1$ and $q = \frac{1}{2}$. (i) μ against ζ_w: (*a*) from (5.7) and (5.8); (*b*) from the first two terms of (5.12). (ii) g_0 against μ (from (5.7) and (5.8)).

Conservation of mass and (5.9) imply that

$$3\sqrt{2}\,k-(162p_0)^{\frac{1}{4}}\int_0^{\infty}(1-\hat{g})\,\mathrm{d}\eta = 1 \tag{5.11}$$

and that $\zeta_w = (162p_0)^{-\frac{1}{4}}$ to leading order.

This suggests that for large ζ_w we expand p_0 and k as

$$\left.\begin{aligned} k &= k_0+k_1\zeta_w^{-1}+\ldots, \\ p_0 &= \tfrac{1}{162}\zeta_w^{-4}+p_{01}\zeta_w^{-5}+\ldots \end{aligned}\right\} \tag{5.12}$$

Equation (5.11) and the wall relationship $\zeta_w = k(2/p_0)^{\frac{1}{4}}$, which must hold exactly, can be used to determine the k_i and p_{0i} in (5.12). In particular, we find $k_0 = 1/(3\sqrt{2})$ (as above), $k_1 = 1-\sqrt{\frac{2}{3}}$, and $p_{01} = 4(1-\sqrt{\frac{2}{3}})/(27\sqrt{2})$. We have assumed here that higher-order terms in the boundary-layer expansion are insignificant to the order calculated in (5.12). It is easily shown that if $\hat{g}$ is expanded in terms of ζ_w^{-1} then the leading-order term is given by (5.10) and that the values of k_0, k_1 and p_{01} are as given. Hence there is no serious omission to the results presented.

The tube law implies that $y_w = kp_0^{-\frac{1}{2}}(X_0-X)$ locally, and hence that the slope of the channel walls will become infinite as $\mu \rightarrow 18$ and $p_0 \rightarrow 0$.

Figure 6 displays the values of g_0 and ζ_w against μ as calculated from (5.7) and

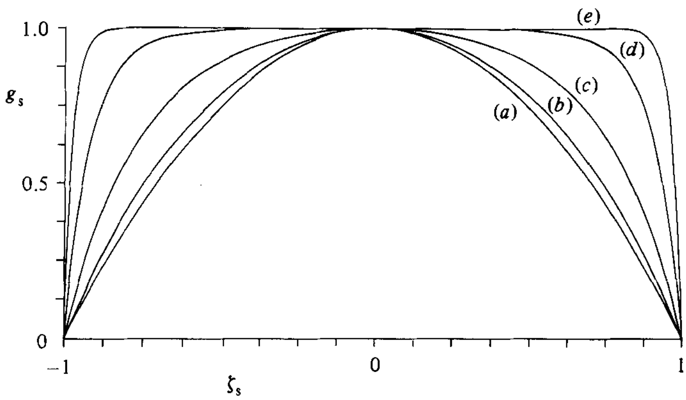

FIGURE 7. Velocity profiles for $q = \frac{1}{2}$, $\epsilon = 1$ and various μ (from (5.6)) where $g_s = g/g_0$ and $\zeta_s = \zeta/\zeta_w$. (a) $\mu = 0.1$; (b) 4.0; (c) 10.0; (d) 15.0; (e) 17.0.

(5.8), and the theoretical values of μ obtained from the first two terms for k in (5.12). Agreement is excellent in the region for which $1-g_0$ is small, i.e. for $\mu < 13$ approximately. The shape of the velocity profile for various μ (from 5.6) is shown in figure 7. Clearly $g(\zeta)$ takes the predicted mainly inviscid form as $\mu \rightarrow 18-$. Also, the velocity is close to parabolic for $\mu \leqslant 1$ approximately, a result easily recovered by expanding for small g_0.

Figure 8 shows results for the full problem (1.1)–(1.5) for various μ with $\epsilon = 1$ and $q = \frac{1}{2}$. A detailed comparison showed excellent agreement between these values (when they could be obtained), and the solution of the local problem (5.1)–(5.4), and it is clear that this self-similar solution provides a valid local description of the flow for $\mu = O(1)$. As expected, X_0 decreases as μ increases. The numerical solution of (1.1)–(1.5) failed for μ greater than about 13, a value consistent with $1-g_0 \ll 1$ (see figure 6) and the solution approaching the limit form.

Breakdown of the solution as $\mu \rightarrow 18$

We have found a numerical solution of (5.2)–(5.4) valid for $\mu < 18$ and a limit solution valid as $\mu \rightarrow 18$. Since $g_0 \leqslant 1$ it follows from (5.7) and (5.8) that $k(2/p_0)^{\frac{1}{4}} > \frac{1}{3}(2p_0)^{\frac{1}{4}}$, which implies that $\mu < 18$. Hence the completely forward flow cannot provide a solution of (5.2)–(5.4) for $\mu > 18$ (nor can any of the other flows – see below). Also, (5.12) implies that the limit solution is valid only as $\mu \rightarrow 18$ from below if $\zeta_w > 0$. An unsuccessful attempt was made to find an alternative limit structure (Tutty 1982; Appendix 1).

The self-similar form (5.1) arises naturally from mass conservation and a balance of the viscous and inertial forces. We could assume instead that the inertia dominates the viscous diffusion as $y_w \rightarrow 0$, so that the flow is governed by the classical boundary-layer equations (1.4) without the diffusion term. However, it is easily shown that as $p \rightarrow -\infty$ the solution (Cole & Aroesty 1968) is valid only for $\mu = 18$, and that the leading-order term is, not surprisingly, the inviscid core flow of the limit solution given above.

The validity of the solution for $\mu < 18$, and the consistency between the numerical and analytical results (both in their failure and agreement) suggests that no steady solution exists for the problem as formulated. It is possible that the difficulties may be resolved by using a shorter axial lengthscale, or by upstream influence. Tutty (1982) considered these, with respect to the problem with $\mu \gg 1$, and they did not

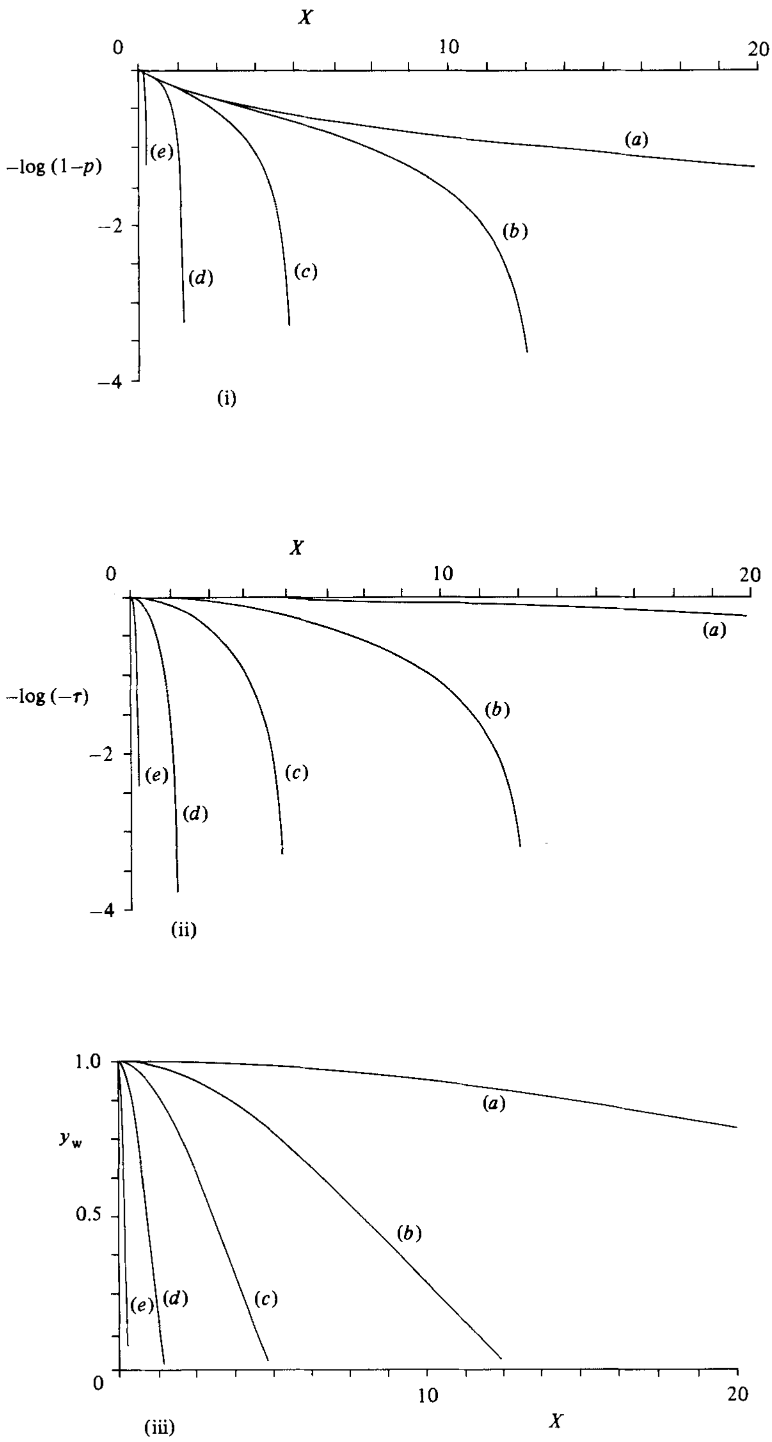

FIGURE 8. (i) Pressure p (on a logarithmic scale). (ii) Upper-wall skin friction τ (on a logarithmic scale). (iii) Channel half-width y_w. Results against distance for a channel with $\epsilon = 1$, $q = \frac{1}{2}$. The asymptotic solutions are graphically identical (where valid). (*a*) $\mu = 0.1$; (*b*) 0.4; (*c*) 1.0; (*d*) 3.0; (*e*) 10.0.

appear to provide a reasonable explanation of the failure of the solution, or a satisfactory flow structure. Alternatively, it may be that the channel–fluid system is inherently unstable and unsteady effects must be included. Some experimental and theoretical evidence supports the latter conjecture (see §9).

Self-similar solutions with reversed flow

Following the above, we can construct solutions of (5.2) which satisfy no slip and have an arbitrary number of regions of reversed flow, including possibly regions of reversed flow adjacent to the walls (for details see Tutty 1982). Such solutions have a limit form of $g = 1$ over most of the channel, with the appropriate reversed jets. Mass conservation is satisfied by this limit form, and there is a range of $g_0 < 1$ for which any such solution can satisfy (5.4). For any particular solution requiring $0 < \zeta_w < \infty$ implies that $\mu < 18$. Hence if $\mu \geqslant 18$ there is no local solution with the self-similar form (5.1). We note that, in common with our problem, a solution of the Jeffery–Hamel problem does not exist for all values of the parameters (Rosenhead 1940).

No concavities or reversals were found in any numerical solutions of (1.1)–(1.5). This, and the excellent agreement between the solutions of (5.1)–(5.4) and (1.1)–(1.5), strongly suggests that, when it exists, the correct local solution is the completely forward flow.

The structure as $X \to X_0$

With $q < \frac{1}{2}$ the inertial terms from the boundary-layer approximation, the minor terms from the expansion of the tube law, and the lower-order terms from the Navier–Stokes equations are ignored in the analysis for $-\mu p \gg 1$. However, none of these appear to force a breakdown of the approximation as $X \to X_0$. Indeed, the condition for such a breakdown seems to be $q > \frac{1}{2}$. With $q = \frac{1}{2}$ only two of these factors are omitted, the boundary-layer equations (1.4) being satisfied exactly. To simplify matters further, consider the problem in which the tube law is given exactly by (2.1) downstream of some point $\hat{X}$ where $p = \hat{p} < 0$. Then, for $-p \gg 1$, the stream function and the pressure have the form (5.1), where $f(\zeta)$ and $p_0(\zeta)$ are power series in Re^{-2}. To leading order f and p_0 are those given above. The lower-order terms give rise to systems of third-order ordinary differential equations with straightforward boundary conditions. In general, forcing terms ensure non-trivial solutions of these problems.

Although, as yet, solutions have not been computed for these lower-order problems, it seems that there will be little difficulty in doing so, and hence that there will be no breakdown in the flow structure as $X \to X_0-$, at least for this model problem.

6. Channels with $q > \frac{1}{2}$

Suppose that $q > \frac{1}{2}$, $\epsilon = 1$ and $\mu = O(1)$. As $y_w \to 0$ the inertial force in the fluid dominates the pressure gradient, and the local problem does not appear to be well posed in the sense that the wall position (and therefore the boundary conditions) depend on the pressure but the pressure does not appear in the governing equations at leading order. If a balance between the viscous and inertial forces is assumed then y_w is $O(X_0 - X)$, and it can be shown (cf. $q = \frac{1}{2}$) that there is no self-similar solution for the local problem that can satisfy both no slip at the walls and mass conservation.

Alternatively, if inertia dominates, the flow must be non-symmetric unless there is a concavity in the flow field at $y = 0$, and unless the transverse velocity is unidirectional there must be reversed flow over part of the channel. Although concavities have been reported for flows through constricted tubes (Smith 1979; Despande, Giddens & Mabon 1976; Forrester & Young 1970), there was no evidence in the numerical solutions (as far as they could be obtained) for any concavity here. We could not find a solution for any self-similar flow with $p \propto (X_0 - X)^{-n}$ and $nq < 1$, a form that might be expected from the results for $q \leqslant \frac{1}{2}$.

The numerical procedure applied to the full problem (1.1)–(1.5) diverged when the walls moved a finite distance into the channel. Hence no complete numerical or analytical solution was found for the problem with $q \geqslant \frac{1}{2}$. This is true also for the problems with $\mu \ll 1$ (§7) and $\mu \gg 1$ (Tutty 1982).

7. Channels with a low compliance

Suppose that $\mu \ll 1$. The tube law suggests the scaling $p = \mu^{-1}\tilde{P}$, and in turn $x = \mu^{-1} Re\, \tilde{X}$. The flow is viscous, the pressure independent of y to leading order, and the solution has the lubrication form (3.1*a*) and (3.2) with $G(\tilde{X}) = y_w^{-3}$, $\zeta = y$ and $p_0 = m = 1$, with $\tilde{P}$ determined by $\tilde{P}' = -G$. Clearly, if $y_w \rightarrow 1-\epsilon > 0$ as $\tilde{P} \rightarrow -\infty$ this solution takes the expected Poiseuille form in the far-downstream limit.

Suppose now that $\epsilon = 1$. There is a singularity at a finite value $\tilde{X}_0$ of $\tilde{X}$ for $q > \frac{1}{3}$, with an asymptotic structure for $-\tilde{P} \gg 1$ similar to that given above. If $q \leqslant \frac{1}{2}$ the flow remains viscous in nature throughout the collapse, and if $\frac{1}{3} < q \leqslant \frac{1}{2}$ the asymptotic structure appears to be valid until the singularity is reached, as for $\mu = O(1)$. If $q > \frac{1}{2}$ then the viscous and inertial forces and the pressure gradient are all of the same order a distance $O(\mu^{(3q-1)/(2q-1)})$ from the singularity. In this region, the governing equations are the nonlinear boundary-layer equations (1.4), and the boundary conditions are no slip at the walls and matching to the incoming flow. It follows from the results of §6 that we do not have an asymptotic solution for this local problem as $y_w \rightarrow 0$, nor a numerical solution for $y_w \ll 1$. Thus, for a channel obeying (1.5), we do not have a complete solution for any value of $\mu \leqslant O(1)$ if $q > \frac{1}{2}$. This is true also for $\mu \gg 1$ (Tutty 1982).

8. Axisymmetric pipe flow

Consider a circular pipe of radius a upstream from a compliant pipe. Non-dimensionalizing on the radius and matching to the incoming Poiseuille flow, as for the channel, gives a uniform pressure across the pipe and the axisymmetric boundary-layer equations as the governing equations. We take as tube law (1.2) and (1.5) with y_w replaced by r_w, where $r = r_w$ is the wall position and r is the non-dimensional radial coordinate.

Suppose that $\mu = O(1)$ and $\epsilon = 1$. The analysis for $-\mu p \leqslant O(1)$ is similar to that outlined in §2 for symmetric channels. When $-\mu p \gg 1$, $r_w = k(-p)^{-q}$ to leading order, where $k = \mu^{-q}$. It follows that if $q < \frac{1}{4}$ the flow is similar to that for a channel with $q < \frac{1}{3}$ and $-\mu p \gg 1$, i.e. viscous-dominated with a parabolic axial velocity profile and the collapse extending indefinitely far downstream. If $q > \frac{1}{4}$ then the inertial force must dominate the pressure gradient as $p \rightarrow -\infty$, as for a channel with $q > \frac{1}{2}$ (§7). No solution was found for this local problem with $q > \frac{1}{4}$.

If $q = \frac{1}{4}$ the inertial force in the axial direction and the pressure gradient are of the same order when $-\mu p \gg 1$. Balancing with the viscous force implies that $p \to -\infty$ exponentially in $X = Re^{-1} x$ as $X \to \infty$, which suggests the self-similar form

$$\left.\begin{aligned} \psi &= (4b)^{-1} f(\zeta), \\ p &= -e^{4bX}, \end{aligned}\right\} \tag{8.1}$$

where $\psi(X, r)$ is the stream function $\zeta = (4b)^{\frac{1}{2}}\, r e^{bX}$ and b is a constant to be determined. The axial momentum equation and (8.1) produce

$$g'' + \frac{g'}{\zeta} + 1 - \tfrac{1}{2} g^2 = 0, \tag{8.2}$$

where $g = f'/\zeta$ gives the axial velocity component. The tube law has the local form

$$\zeta_w = k(4b)^{\frac{1}{2}}, \tag{8.3}$$

where $k = \mu^{-\frac{1}{4}}$. The boundary conditions are that g is regular in $\zeta < \zeta_w$ and

$$\left.\begin{aligned} g(\zeta_w) &= 0, \\ \int_0^{\zeta_w} \zeta g(\zeta)\, d\zeta &= \tfrac{1}{4} b, \end{aligned}\right\} \tag{8.4}$$

from no slip at the wall and mass conservation respectively.

Unfortunately (8.2) cannot be integrated directly, and an analysis similar to that for (5.2) is not possible. However, we can show that a solution of (8.2)–(8.4) can exist only if $\mu < 128$, and that if a solution exists when $0 < 128 - \mu \ll 1$ then the flow must be essentially uniform across the pipe, the viscous effects being restricted to thin layers (cf. the channel with $q = \frac{1}{2}$ and $0 < 18 - \mu \ll 1$).

From (8.2) we see that any turning point at which $g > \surd 2$ must be a minimum. Hence if g and g' are continuous and $g > \surd 2$ at any point, then either g is monotonic or $g > \surd 2$ at all points. Thus $\surd 2$ is the maximum g can take if no slip at the wall is to be satisfied and the solution is to be smooth. It follows that $\int_0^{\zeta_w} \zeta g(\zeta)\, d\zeta < \zeta_w^2/\surd 2$, and hence, from (8.4), that mass conservation can be satisfied only if $k > k_0$, where $k_0 = 128^{-\frac{1}{4}}$; that is, if $\mu < 128$. Further, if $k - k_0 \ll 1$ then $\surd 2 - g \ll 1$ over most of the tube $(0 \leqslant \zeta \leqslant \zeta_w)$. Thus as $\mu \to 128-$ any solution of (8.2) and (8.3) must have an essentially inviscid uniform mainstream with a viscous boundary layer at the wall. Daniels & Eagles (1978) have studied (8.2) for pipes with fixed exponential walls. They found a number of solutions, including a limit solution with basically uniform flow, and solutions with regions of reversed flow, all of which would be expected from the above and the results of §5.

We note that, unlike the analogous channel case, the solution remains formally valid as $\mu - 128 \to 0-$ (see Daniels & Eagles 1978). This difference, like the existence of the singularity for channels but not for pipes, occurs because with a pipe the velocity varies as the inverse of the radius squared, whereas for a channel the velocity varies as the inverse of the width.

9. Comparison with experimental results

In experimental studies of flow in collapsible tubes 'self-excited oscillations' are commonly observed. If the outlet of the tube is substantially collapsed and the flow rate Q is large enough, then the fluid–tube system can spontaneously become unstable and cannot be regarded as quasi-steady. Brower & Scholten (1975) report that 'for a given tube, there seems to be a critical flow, characteristic only of the collapsible tube, which determines when oscillations are initiated'. Further, they propose that this instability occurs when the mean fluid velocity in the tube exceeds the 'phase velocity' (the speed of propagation of a small pressure disturbance along the elastic tube) or, more precisely, the low-frequency limit of the phase velocity. This proposed connection between the fluid and phase velocities and the stability of the system has also appeared in theoretical studies (Oates 1975; Shapiro 1977*a*; Kamm & Shapiro 1979).

In our terms, Brower & Scholten's proposals become

(i) for a given tube there is a critical value of μ, μ_c say, such that the fluid–tube system will spontaneously become unstable for $\mu \geqslant \mu_c$;

(ii) μ_c is the minimum value of μ for which the mean fluid velocity can match the phase velocity at any stage of the collapse.

In most attempts to calculate the phase velocity of an elastic tube filled with a viscous fluid it is assumed that the phase velocity is large compared with the mean fluid velocity and hence that the Navier–Stokes equations can be linearized (see e.g. Womersley 1955). We found no studies on the main problem of interest here, that of a viscous flow with the fluid and phase velocities of the same order. For an inviscid fluid with uniform flow in an elastic tube obeying a tube law the phase velocity is given by

$$C = \left(A\frac{\mathrm{d}p}{\mathrm{d}A}\right)^{\frac{1}{2}}, \tag{9.1}$$

where c is the phase velocity (relative to the fluid velocity) and A is the nondimensional cross-sectional area of the tube (Shapiro 1977*a*).

For a steady inviscid uniform flow in an elastic tube with tube law $A = A(p)$ the equations of motion reduce to

$$\frac{\mathrm{d}u}{\mathrm{d}x}\left(A - u^2\frac{\mathrm{d}A}{\mathrm{d}p}\right) = 0. \tag{9.2}$$

Mass conservation requires $Au = Q$, where Q is a constant. Hence, if $\mathrm{d}u/\mathrm{d}x$ is non-zero,

$$\frac{\mathrm{d}A}{\mathrm{d}p} = \frac{A^3}{Q^2}, \tag{9.3}$$

the solution of which is

$$A(p) = \left(B - 2\frac{p}{Q^2}\right)^{-\frac{1}{2}}, \tag{9.4}$$

where $B = 1/A^2(0)$ is a constant. Thus we see that an axially varying one-dimensional steady flow of an inviscid, incompressible fluid can exist in a collapsible tube with tube law $A = A(p)$ only if $A(p)$ has the form (9.4), and that the fluid velocity must necessarily match the phase velocity of the system.

Consider now the channel with $\epsilon = 1$ and $S(\mu p)$ given by (1.5). Then as $p \to -\infty$ the tube law will take the form (9.4) if $q = \frac{1}{2}$ and $\mu = 18$. In this case we also have an essentially uniform inviscid flow (provided the Reynolds number is sufficiently large). With $q = \frac{1}{2}$, $\bar{u} < c$ throughout the collapse if and only if $\mu < 18$, where $\bar{u}$ is the mean fluid velocity and c is given by (9.1). Similarly, the flow is basically inviscid and the tube law matches (9.4) for a pipe with $\epsilon = 1$, $q = \frac{1}{4}$ and $\mu = 128$, with $\bar{u} < c$ throughout the collapse if and only if $\mu < 128$.

Thus our results are consistent with (i) and (ii) above in that a complete steady solution was found for the cases with $\bar{u}/c < 1$ throughout the collapse ($q < \frac{1}{2}$ for the channel, $q < \frac{1}{4}$ for the pipe), no complete solution was found for the cases such that $\bar{u}/c \to \infty$ as $p \to -\infty$ ($q > \frac{1}{2}$ for the channel, $q > \frac{1}{4}$ for the pipe), and for cases in which the tube law takes the form (9.4) a solution was found only for μ such that $\bar{u}/c < 1$ throughout the collapse.

10. Final remarks

The analytical solutions presented above have a reasonably simple form, and showed a pleasing agreement with the numerical solutions. Perhaps the most interesting feature of the results is not the solutions as such, but where they failed. We have found a singularity in the solution for some channels such that the channel width tends to zero at a finite distance downstream. This singularity is unusual in that the boundary-layer equations seem to apply (except for $q = \frac{1}{2}$ with $18-\mu \ll 1$) and the flow structure does not break down as the singularity is approached, i.e. although physically impossible the solution is mathematically valid until the singularity is reached. This singularity may be admitted only because the tube law has a very simple form, and it may not occur with a more realistic model of wall elasticity. Also, it may be possible that an 'elastic jump' – a shock-like change in the system where the tube expands markedly on a relatively short lengthscale – could provide a continuation to the present solution. However, there are several reasons for doubting the latter. First, the ratio of streamwise to crosswise lengthscale is large and remains so as the singularity is approached; Secondly, in all known elastic jumps the flow is supercritical ($\bar{u}/c > 1$) upstream and subcritical ($\bar{u}/c < 1$) downstream, and, in particular, this must be so in an 'inviscid' jump (Oates 1975; Cowley 1982). We seem to require a subcritical to subcritical jump if the flow is to remain steady. These points are most easily seen from the model problem presented at the end of §5. In particular, the ratio of streamwise to crosswise scaling is constant, and $\bar{u}/c$ is always less than one and tends to a constant as the singularity is approached. Of course, this argument is in no way rigorous. In particular, we do not have an expression for c that is valid for viscous flows.

We have found encouraging agreement between the experimental results of Brower & Scholten (1975) (and others: see Pedley 1980, chap. 6) and the limitation on our solution when the cross-sectional area behaves as $(-p)^{-\frac{1}{2}}$. It is stressed that we have not proved that the system must be unsteady if $\bar{u}/c > 1$, nor that Brower & Scholten's hypothesis (i and ii, §9) are valid. However the agreement does suggest that the addition of time dependence and a stability analysis of our solutions may be a profitable extension to the present theory.

Finally, we note that models of flow in collapsible tubes have been formulated with Poiseuille flow assumed valid locally (e.g. Rubinow & Keller 1972; Wild *et al.* 1977), and that our study supports the use of these models in certain circumstances; in particular, when the system is (quasi-)symmetric, the cross-sectional area A behaves like $(-p)^{-q}$ with $q \leqslant \frac{1}{2}$, $\mathrm{d}A/\mathrm{d}p \leqslant O(1)$, and the incoming flow is high-Reynolds-number Poiseuille flow.

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